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On the interior regularity of weak solutions to the non-stationary Stokes system

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Abstract In this paper, we prove that any weak solution to the non-stationary Stokes system in 3D with right hand side $-\mathrm{div}\,\mathbf{f}$ satisfying (1.4) below, belongs to $C(\,]0, T[;\, \mathbf{C}^{\alpha}(\Omega))$. The proof is based on Campanato-type inequalities and the existence of a local pressure introduced in Wolf [13].

Keywords Non-stationary Stokes system · Interior regularity

Mathematics Subject Classification (2000) 35Q30 · MSC 35D10

1 Introduction. Statement of main result

Let Ω be a bounded domain in $\mathbb{R}^3$, let $0 < T < \infty$ and define $Q := \Omega\times]0, T[$. We consider the Stokes system

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \nabla p \;=\; -\mathrm{div}\,\mathbf{f} \quad \text{in } Q, \tag{1}$$

$$\mathrm{div}\,\mathbf{u} \;=\; 0 \quad \text{in } Q, \tag{2}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ denotes the velocity vector, p the pressure and $-\mathrm{div}\,\mathbf{f}$ an external force.

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The aim of the present paper is to study the interior regularity of weak solutions $\mathbf{u}$ of (1), (2) regardless of whether $\mathbf{u}$ satisfies boundary conditions on $\partial\Omega\times]0,T[$, and an initial condition on $\Omega\times\{0\}$.

To define the notion of weak solution of (1), (2) we introduce the following notations. By $W^{m,2}(\Omega)$ $(m=1,2,\dots)$ we denote the usual Sobolev space. If the boundary $\partial\Omega$ is Lipschitz, we define

$$\mathring{W}^{1,2}(\Omega) := \Big\{v\in W^{1,2}(\Omega);\ v=0 \text{ a.e. on } \partial\Omega\Big\}.$$

Throughout the paper, we write $\mathbf{C}^{\alpha}(\Omega) := [C^{\alpha}(\Omega)]^3$, $\mathbf{L}^{q}(\Omega) := [L^{q}(\Omega)]^3$, $\mathbf{W}^{m,2}(\Omega) := [W^{m,2}(\Omega)]^3$ etc. Here

$$C^{\alpha}(\Omega):=\Big\{v:\Omega\to\mathbb{R};\ \sup_{\substack{x,y\in K\\ x\neq y}}\frac{|v(x)-v(y)|}{|x-y|^{\alpha}}<\infty \text{ for every compact } K\subset\Omega\Big\}\ (0<\alpha<1).$$

Next, let

$$\mathbf{W}^{1,2}_{\sigma}(\Omega) := \Big\{\mathbf{w}\in\mathbf{W}^{1,2}(\Omega;\mathbb{R}^3);\ \operatorname{div}\mathbf{w}=0\ \text{ a.e. in } \Omega\Big\},$$
$$\mathring{\mathbf{W}}^{1,2}_{\sigma}(\Omega) := \Big\{\mathbf{w}\in\mathbf{W}^{1,2}_{\sigma}(\Omega);\ \mathbf{w}=0\ \text{ a.e. on } \partial\Omega\Big\}.$$

Given a normed vector space X with norm $\|\cdot\|$, we denote by $L^{s}(0,T;X)$ $(1\le s\le\infty)$ the vector space of all Bochner measurable functions $z:]0,T[\to X$ such that

$$\int_0^T\|z(t)\|^s\,\mathrm{d}t<\infty \quad \text{if } 1\le s<\infty, \qquad \operatorname*{ess\,sup}_{]0,T[}\|z(t)\|<\infty \quad \text{if } s=\infty.$$

(see, e.g., [10; Chap. IV, 1] for details).

Finally define

$$\mathbf{C}^{\infty}_{c,\sigma}(\Omega) := \Big\{\boldsymbol{\varphi}\in\mathbf{C}^{\infty}_{c}(\Omega);\ \operatorname{div}\boldsymbol{\varphi}=0 \text{ in } \Omega\Big\}$$

(throughout the subscript c means that the function under consideration has compact support in its domain of definition).

Definition 1 *Let* $\mathbf{f}\in L^2(0,T;[L^2(\Omega)]^9)$*. The function* $\mathbf{u}\in L^2(0,T;\mathbf{W}^{1,2}_{\sigma}(\Omega))$ *is called a weak solution of (2), (1) if*

$$-\int_Q\mathbf{u}\cdot\boldsymbol{\varphi}_t\,\mathrm{d}x\,\mathrm{d}t+\int_Q\nabla\mathbf{u}:\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t=\int_Q\mathbf{f}:\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t \tag{3}$$

for all $\boldsymbol{\varphi}\in C^{\infty}_{c}(]0,T[;\mathbf{C}^{\infty}_{c,\sigma}(\Omega))$ (in what follows, for $\mathbf{v}=(v_1,v_2,v_3)$ define $\nabla\mathbf{v}=\left\{\frac{\partial v_i}{\partial x_j}\right\}$; for matrices $A=\{A_{ij}\}$, $B=\{B_{ij}\}$ define $A:B=\sum_{i,j=1}^{3}A_{ij}B_{ij}$).

Clearly, for this definition to make sense the weaker assumption $\mathbf{f}\in L^2_{\mathrm{loc}}(0,T;[L^2_{\mathrm{loc}}(\Omega)]^9)$ is sufficient. Moreover, it is readily seen that the proof of our main result can be easily extended to include more general right hand sides of the form $\mathbf{f}_0-\operatorname{div}\mathbf{f}$.

Remark 1 1. Our definition of weak solution of (1), (2) is closely related to the one introduced in [9] for the non-stationary Navier-Stokes system (cf. also the definition in [10; Chap. IV, 2.1]).

2. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Suppose we are given $\mathbf{f} \in L^2(0, T; [L^2(\Omega)]^9)$, and $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$ with $\operatorname{div} \mathbf{u}_0 = 0$ in sense of distributions in Ω. Then there exists a uniquely determined $\mathbf{u} \in L^2(0, T;\ \mathring{\mathbf{W}}_\sigma^{1,2}(\Omega)) \cap C([0, T]; \mathbf{L}^2(\Omega))$ such that $\mathbf{u}(x, 0) = \mathbf{u}_0$ for a.e. $x \in \Omega$, and (3) holds. In addition,

$$\frac{1}{2}\int_\Omega |\mathbf{u}(x,t)|^2\,\mathrm{d}x + \int_0^t\int_\Omega |\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int_\Omega |\mathbf{u}_0(x)|^2\,\mathrm{d}x + \int_0^t\int_\Omega \mathbf{f} : \nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}s \quad \forall t \in [0, T]$$

(cf. [10; Chap. IV, 2.4]). □

To state our main result we need the following notations. For $x_0 \in \mathbb{R}^3$ and $t_0 \in \mathbb{R}$, define

$$B_r(x_0) := \left\{x \in \mathbb{R}^3 \,\middle|\, |x_0 - x| < r\right\}, \quad Q_r(x_0, t_0) := B_r(x_0)\times\,]t_0 - r^2, t_0[\,.$$

Let $E \subset \mathbb{R}^4$ and $F \subset \mathbb{R}^3$ be measurable sets, where $F \times \{t\} \subset E$. Then for $g \in L^1(E)$, define

$$g_E := \frac{1}{\operatorname{meas} E}\int_E g\,\mathrm{d}x\,\mathrm{d}t, \quad g_F(t) := \int_F g(x,t)\,\mathrm{d}x$$

The main result of our paper is the following.

Theorem *Let* $\mathbf{f} \in L^2(0, T; [L^2(\Omega)]^9)$. *Suppose that there exists* $\alpha \in]0, 1[$ *such that*

$$\begin{cases} \textit{for every } Q' = \Omega'\times\,]t', T[\ (\Omega' \textit{ open}) \textit{ with } \overline{Q'} \subset Q \textit{ there exists} \\ c = \textit{const } (\textit{possibly depending on } \operatorname{dist}(\Omega', \partial\Omega) \textit{ and } t') \textit{ such that} \\ \displaystyle\int_{Q_r} |\mathbf{f} - \mathbf{f}_{Q_r}|^2\,\mathrm{d}x\,\mathrm{d}t \le c\,r^{3+2\alpha} \quad \forall \overline{Q}_r \subset Q'. \end{cases} \tag{4}$$

Let $\mathbf{u} \in L^2(0, T; \mathbf{W}_\sigma^{1,2}(\Omega)) \cap C_w(\,]0, T[; \mathbf{L}^2(\Omega))$ *be a weak solution of (1), (2). Then*

$$\mathbf{u} \in C(\,]0, T[\,; \mathbf{C}^\alpha(\Omega)).$$

Remark 2 1. Assume $\mathbf{f} = 0$. Let a be any continuous function in $]0, T[$, and let z be harmonic in Ω. Define $\mathbf{u}(x, t) := (\nabla z(x))a(t)$, $(x, t) \in Q$ (cf. [9]). Clearly, $\mathbf{u} \in C(\,]0, T[\,; \mathbf{C}^\infty(\Omega))$, $\operatorname{div}\mathbf{u} = 0$ in Q. Consider $Q' = \Omega'\times\,]t', t''[$ where Ω' open, $\overline{\Omega'} \subset \Omega$, and $0 < t' < t'' < T$. Then $\mathbf{u}\big|_{Q'}$ is a weak solution of (1), (2) in the sense of the above definition with Q' in place of Q.
Thus, the regularity property of any weak solution of (1), (2) stated in our main result, is the best possible under assumption (4).
2. Global L^q-regularity results for solutions to (1), (2) under zero initial-boundary-conditions are proved by Koch/Solonnikov [4,5] by using methods of potential theory. In these papers, the existence of a pressure $p = p_1 + \dfrac{\partial P}{\partial t}$ is established where $p_1 \in L^q$ and P is harmonic.
3. An interior regularity result for weak solutions of the Navier–Stokes system which satisfy conditions due to Kiselev/Ladyženskaya, has been proved by Ohyama [8]. Serrin [9] improved this result by showing that any weak solution $\mathbf{u}$ to the Navier–Stokes system with $\mathbf{u} \in L^\infty(0, T; \mathbf{L}^2(\Omega))$, $\operatorname{rot}\mathbf{u} \in L^2(0, T; \mathbf{L}^2(\Omega))$ and which satisfies the additional condition $\mathbf{u} \in L^s(0, T; \mathbf{L}^q(\Omega))\left(\dfrac{3}{q} + \dfrac{2}{s} < 1,\ 3 < q < \infty\right)$, is of class $\mathbf{C}^\infty$ in space

variables (cf. [10; Chap. V, 1.8] for a global result of this type). A refinement of Serrin's result has been developed by Takahashi [11].

The proof of our main result is based on arguments of Campanato's regularity theory of weak solutions of parabolic systems, and a recently obtained result by Wolf [13] about the existence of a pressure $p = p_0 + \dfrac{\partial p_h}{\partial t}$, where $p_0 \in L^2(t', t''; L^2(\Omega'))(\Omega' \subset \overline{\Omega}' \subset \Omega, 0 < t' < t'' < T)$ and $x \mapsto p_h(x, t)$ is harmonic in Ω' for all $t \in]t', t''[$. Here the assumption $\mathbf{u} \in C_w(]0, T[; \mathbf{L}^2(\Omega))$ implies $p_h \in C(]0, T[; W^{m,2}(\Omega'))$ $(m = 1, 2, \ldots, \Omega' \subset \overline{\Omega'} \subset \Omega)$. In contrast to [4,5], the existence of this pressure also applies to nonlinear situations.

Our paper is organized as follows. In Sect. 2, we prove a Campanato-type inequality for any weak solution of (1), (2) Section. 3 is devoted to the proof of the interior Hölder continuity of a weak solution of (1), (2) provided the existence of a pressure in $L^2_{\mathrm{loc}}(Q)$ is known. Finally, in Sect. 4 we complete the proof of our main theorem by using the local pressure from [13].

For reader's convenience, in the appendix we present the tools used in Sects. 2 and 3.

2 Campanato-type inequalities for u

The following result will be obtained by using ideas of Campanato [1], where we have to construct a special divergence-free test function.

We note that Proposition 1 and 2, and above all the decomposition theorem stated in the appendix, form the basis for our proof of the main result.

Proposition 1 *Let* $\mathbf{f} \in L^2(0, T; [L^2(\Omega)]^9)$. *Let* $\mathbf{u} \in L^2(0, T; \mathbf{W}^{1,2}_\sigma(\Omega)) \cap C_w(]0, T[; \mathbf{L}^2(\Omega))$ *be a weak solution of* (2), (1). *Fix any* $Q_{R_0} = B_{R_0} \times]t_0 - R_0^2, t_0[$ *such that* $\overline{Q}_{R_0} \subset Q$. *Let* $0 < R \le R_0$. *Then, the following Campanato-type inequality holds:*

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{]t_0 - r^2, t_0[} \int_{B_r} |\mathbf{u}(x,t) - \mathbf{u}_{B_r}(t)|^2 \,\mathrm{d}x + \int_{Q_r} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le c\Big(\frac{r}{R}\Big)^5 \Bigg(\operatorname*{ess\,sup}_{]t_0 - R^2, t_0[} \int_{B_R} |\mathbf{u}(x,t) - \mathbf{u}_{B_R}(t)|^2 \,\mathrm{d}x + \int_{Q_R} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \Bigg) \\
&\quad\quad + c \int_{Q_R} |\mathbf{f} - \mathbf{f}_{Q_R}|^2 \,\mathrm{d}x\,\mathrm{d}t \quad \forall\, 0 < r \le R,
\end{aligned}
\tag{5}
$$

where the constant $c = c(R_0)$ *does not depend on* r *and* R.

Proof We proceed in four steps. To begin with, we note that there holds

$$
-\int_{Q_R} \mathbf{u} \cdot \boldsymbol{\varphi}_t \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_R} \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t = \int_{Q_R} \mathbf{f} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \tag{6}
$$

for all $\boldsymbol{\varphi} \in C^\infty_c(]t_0 - R^2, t_0[; \mathbf{C}^\infty_{c,\sigma}(B_R))$.

(1) *Decomposition of* $\mathbf{u}$ *in* Q_R. First, there exists a uniquely determined

$$
\mathbf{w} \in L^2(t_0 - R^2, t_0; \mathring{\mathbf{W}}^{1,2}_\sigma(B_R)) \cap C([t_0 - R^2, t_0]; \mathbf{L}^2(B_R))
$$

with the following properties: $\mathbf{w}(t_0 - R^2, x) = 0$ for a.e. $x \in B_R$,

$$
-\int_{Q_R} \mathbf{w} \cdot \boldsymbol{\varphi}_t \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_R} \nabla \mathbf{w} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t = \int_{Q_R} \mathbf{f} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \tag{7}
$$

for all $\varphi \in C_c^\infty(]t_0 - R^2, t_0[; \mathbf{C}_{c,\sigma}^\infty(B_R))$, and

$$\|\mathbf{w}\|^2_{C([t_0-R^2,t_0];\mathbf{L}^2(B_R))} + \int_{Q_R} |\nabla \mathbf{w}|^2 \,\mathrm{d}x\,\mathrm{d}t \leq c \int_{Q_R} |\mathbf{f} - \mathbf{f}_{Q_R}|^2 \,\mathrm{d}x\,\mathrm{d}t \tag{8}$$

(c = const independent of R) (cf., e.g., [10,12]).

Now define $\mathbf{v} := \mathbf{u} - \mathbf{w}$. Then $\mathbf{v} \in L^2(t_0 - R^2, t_0; \mathbf{W}_\sigma^{1,2}(B_R)) \cap L^\infty(t_0 - R^2, t_0; \mathbf{L}^2(B_R))$. By (6) and (7),

$$-\int_{Q_R} \mathbf{v} \cdot \varphi_t \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_R} \nabla \mathbf{v} : \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{9}$$

for all $\varphi \in C_c^\infty(]t_0 - R^2, t_0[; \mathbf{C}_{c,\sigma}^\infty(B_R))$.

We now introduce new variables:

$$y := \frac{x - x_0}{R}, \quad s := \frac{t - t_0}{R^2}, \quad (x, t) \in Q_R.$$

Define

$$\mathbf{V}(y, s) := \mathbf{v}(x_0 + Ry, t_0 + R^2 s) \quad \text{for a.e.} \quad (y, s) \in Q_1$$

(recall $Q_1 = Q_1(0, 0) = B_1 \times]-1, 0[$). It follows $\mathbf{V} \in L^2(-1, 0; \mathbf{W}_\sigma^{1,2}(B_1)) \cap L^\infty(-1, 0; \mathbf{L}^2(B_1))$, and

$$-\int_{Q_1} \mathbf{V} \cdot \boldsymbol{\phi}_t \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_1} \nabla \mathbf{V} : \nabla \boldsymbol{\phi} \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{10}$$

for all $\boldsymbol{\phi} \in C_c^\infty(]-1, 0[; \mathbf{C}_{c,\sigma}^\infty(B_1))$.

(2) *Interior differentiability of* $\mathbf{V}$. Let $-1 < s_0 < 0$. Let be $\eta \in C^\infty(\mathbb{R})$ with $\mathrm{supp}(\eta) \subset]-1, s_0[$, and let be $\boldsymbol{\psi} \in \mathbf{C}_{c,\sigma}^\infty(B_1))$. Fix any s_1 such that $s_0 < s_1 < 0$. Then the function

$$\boldsymbol{\phi}(y, s) := \frac{1}{\lambda} \boldsymbol{\psi}(y) \int_{s-\lambda}^{s} \eta(\tau)\,\mathrm{d}\tau, \quad (y, s) \in Q_1, \quad 0 < \lambda < s_1 - s_0$$

is admissible in (10). We obtain

$$\int_{B_1} \frac{\partial \mathbf{V}_\lambda}{\partial t}(y, s) \cdot \boldsymbol{\psi}(y)\,\mathrm{d}y + \int_{B_1} \nabla \mathbf{V}_\lambda(y, s) : \nabla \boldsymbol{\psi}(y)\,\mathrm{d}y = 0 \tag{11}$$

for a.e. $s \in]-1, s_0[$ (note that the null set in $]-1, s_0[$ where this identity fails, does not depend on $\boldsymbol{\psi}$). Here

$$\mathbf{V}_\lambda(y, s) := \frac{1}{\lambda} \int_s^{s+\lambda} \mathbf{V}(y, \tau)\,\mathrm{d}\tau, \quad (y, s) \in]-1, s_0[\times B_1$$

denotes the well-known Steklov mean of $\mathbf{V}(y, \cdot)$.

Next, let $\boldsymbol{\psi}^\varepsilon(y) := \int_{B_1} \omega_\varepsilon(y-z)\boldsymbol{\psi}(z)\,\mathrm{d}z$, $0 < \varepsilon < \frac{1}{2}\mathrm{dist}(\mathrm{supp}(\boldsymbol{\psi}), \partial B_1)$, $y \in \mathbb{R}^3$, denote the mollification of $\boldsymbol{\psi}$, where $\omega_\varepsilon(\xi) = \frac{1}{\varepsilon^3}\omega\left(\frac{\xi}{\varepsilon}\right)$, $\xi \in \mathbb{R}^3$, ω = the standard mollifying kernel. We insert $\boldsymbol{\psi}^\varepsilon$ in (11) in place of $\boldsymbol{\psi}$ and shift the mollification from $\boldsymbol{\psi}^\varepsilon$ to $\mathbf{V}_\lambda$ by the aid of Fubini's theorem. This gives (11) with $\mathbf{V}_\lambda^\varepsilon$ in place of $\mathbf{V}_\lambda$. Then we insert the function

$$\boldsymbol{\psi}(y) = \mathrm{rot}\left[\mathrm{rot}\,\mathbf{V}_\lambda^\varepsilon(y, s)\zeta^2(y)\eta^2(s)\right], \quad (y, s) \in Q_1,$$

into (11) where ζ and η are appropriate cut-off functions for B_1 and $]-1, s_0[$, respectively. Then by a routine argument we obtain

$$\operatorname*{ess\,sup}_{]-(\frac{1}{4})^2,0[} \|\nabla \mathbf{V}(s)\|^2_{[\mathbf{W}^{2,2}(B_{1/4})]^9} \,\mathrm{d}y$$
$$\leq c\left(\operatorname*{ess\,sup}_{]-1,0[} \int_{B_1} |\mathbf{V}(y,s) - \mathbf{V}_{B_1}(s)|^2 \,\mathrm{d}y + \int_{Q_1} |\nabla \mathbf{V}|^2 \,\mathrm{d}y\,\mathrm{d}\tau \right) \tag{12}$$

(3) *Campanato-type inequality for* $\mathbf{V}$. Let $0 < \rho \leq \frac{1}{4}$. Following [1,2] we combine Poincaré's inequality and Sobolev's imbedding theorem $W^{2,2}(B_\rho) \subset C(\overline{B}_\rho)$ to obtain

$$\operatorname*{ess\,sup}_{]-\rho^2,0[} \int_{B_\rho} |\mathbf{V}(y,s) - \mathbf{V}_{B_\rho}(s)|^2 \,\mathrm{d}y + \int_{Q_\rho} |\nabla \mathbf{V}|^2 \,\mathrm{d}y\,\mathrm{d}s$$
$$\leq c\rho^5 \left(\operatorname*{ess\,sup}_{]-1,0[} \int_{B_1} |\mathbf{V}(y,s) - \mathbf{V}_{B_1}(s)|^2 \,\mathrm{d}y + \int_{Q_1} |\nabla \mathbf{V}|^2 \,\mathrm{d}y\,\mathrm{d}s \right)$$

This inequality is trivially true for $\frac{1}{4} < \rho \leq 1$.

(4) *Concluding the proof*. Returning from $\mathbf{V}$ to $\mathbf{v}$ gives

$$\operatorname*{ess\,sup}_{]t_0-r^2,t_0[} \int_{B_r} |\mathbf{v}(x,t) - \mathbf{v}_{B_r}(t)|^2 \,\mathrm{d}x + \int_{Q_r} |\nabla \mathbf{v}|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq c\left(\frac{r}{R}\right)^5 \left(\operatorname*{ess\,sup}_{]t_0-R^2,t_0[} \int_{B_R} |\mathbf{v}(x,t) - \mathbf{v}_{B_R}(t)|^2 \,\mathrm{d}x + \int_{Q_R} |\nabla \mathbf{v}|^2 \,\mathrm{d}x\,\mathrm{d}t \right)$$

for all $0 < r \leq R$ ($c =$ const independent of r) (recall $B_r = B_r(x_0)$, $Q_r = B_r(x_0) \times]t_0 - r^2, t_0[$).

Combining this result and (8) we obtain (5). □

Remark To get rid from the pressure in the Stokes (resp. Navier–Stokes) equations, it is standard to apply the operator rot to each term of these equations (see, e. g., [6,8,9,12]). We have sketched this procedure when starting from (10).

Another aspect of this procedure is as follows. Using our above notations, from (10) we obtain

$$\int_{-1}^{s_0} \int_{B_1} (\mathrm{rot}\, \mathbf{V}^\varepsilon_\lambda)_t \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t + \int_{-1}^{s_0} \int_{B_1} (\nabla \mathrm{rot}\, \mathbf{V}^\varepsilon_\lambda) \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t = 0$$

for any $\boldsymbol{\varphi} \in C^\infty_c(]0, T[\,; \mathbf{C}^\infty_c(B_1))$, where $0 < \lambda < s_1 - s_0$, and $\varepsilon > 0$ is sufficiently small. Since $\mathbf{V} = \lim_{\varepsilon\to 0} \lim_{\lambda\to 0} \mathbf{V}^\varepsilon_\lambda$, it follows $\mathrm{rot}\, \mathbf{V} \in L^\infty(-1, s_0; \mathbf{L}^2(B_\rho)) \cap L^2(-1, s_0; \mathbf{W}^{1,2}(B_\rho))$ $(0 < \rho < 1)$. This argument can be repeated to obtain $\mathrm{rot}\, \mathbf{V} \in C^\infty(]-1, 0[\,; \mathbf{C}^\infty(B_1))$.

Next, each component of $\mathrm{rot}\, \mathbf{V}$ is a solution of the homogeneous heat equation in $B_1 \times]-1, 0[$. It follows

$$\int_{Q_\rho} |\mathrm{rot}\, \mathbf{V} - (\mathrm{rot}\, \mathbf{V})_{Q_\rho}|^2 \,\mathrm{d}x\,\mathrm{d}t \leq c\rho^7 \int_{Q_1} |\mathrm{rot}\, \mathbf{V} - (\mathrm{rot}\, \mathbf{V})_{Q_1}|^2 \,\mathrm{d}x\,\mathrm{d}t$$

for all $0 < \rho < 1$ (cf. [1,2]). As above, returning $\mathbf{V}$ to $\mathbf{v}$ we obtain

$$\int_{Q_r} |\mathrm{rot}\, \mathbf{v} - (\mathrm{rot}\, \mathbf{v})_{Q_r}|^2 \,\mathrm{d}x\,\mathrm{d}t \leq c\left(\frac{r}{R}\right)^7 \int_{Q_R} |\mathrm{rot}\, \mathbf{v} - (\mathrm{rot}\, \mathbf{v})_{Q_R}|^2 \,\mathrm{d}x\,\mathrm{d}t$$

for all $0 < r \le R$. Observing that $\mathbf{u} = \mathbf{v} + \mathbf{w}$ it follows

$$\begin{aligned}
&\int_{Q_r} |\mathrm{rot}\,\mathbf{u} - (\mathrm{rot}\,\mathbf{u})_{Q_r}|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\le c\Big(\frac{r}{R}\Big)^7 \int_{Q_R} |\mathrm{rot}\,\mathbf{u} - (\mathrm{rot}\,\mathbf{u})_{Q_R}|^2\,\mathrm{d}x\,\mathrm{d}t \,+\, c\int_{Q_R} |\mathrm{rot}\,\mathbf{w}|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\le c\Big(\frac{r}{R}\Big)^7 \int_{Q_R} |\mathrm{rot}\,\mathbf{u} - (\mathrm{rot}\,\mathbf{u})_{Q_R}|^2\,\mathrm{d}x\,\mathrm{d}t \,+\, c\int_{Q_R} |\mathbf{f} - \mathbf{f}_{Q_R}|^2\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

(cf. (8); note that $\int_{B_R} |\mathrm{rot}\,\mathbf{w}|^2\,\mathrm{d}x = \int_{B_R} |\nabla\mathbf{w}|^2\,\mathrm{d}x$ for a.e. $t \in]t_0 - R^2, t_0[$).

Inequalities of this type have been obtained by the first named author in his unpublished note: *On some integral estimates on weak solutions of the non-stationary Stokes system*. Preprint, Univ. Catania, 22. Nov. 1989. The techniques used in this note do not give any information about the continuity of $\mathbf{u}$ with respect to t.

The main result of our present paper shows that $\mathbf{u} \in C(\,]0, T[\,; \mathbf{C}^\alpha(\Omega))$. □

3 An integral estimate for the case when a pressure in L^2 exists

The following result is of independent interest. It shows that if $\mathbf{U}$ satisfies the conditions of Prop. 2 for all cylinders $Q_R \subset \overline{Q}' \subset Q$, then it is Hölder continuous in Q' (cf. the discussion at the end of Sect. 4).

Proposition 2 *Let* $\mathbf{f} \in L^2(Q_R; \mathbb{R}^9)$. *Suppose that there exists* $\alpha \in]0, 1[$ *and* $c = \mathrm{const} > 0$ *such that*

$$\int_{Q_r} |\mathbf{f} - \mathbf{f}_{Q_r}|^2\,\mathrm{d}x\,\mathrm{d}t \,\le\, cr^{3+2\alpha} \quad \forall\, 0 < r \le R. \tag{13}$$

Let $\mathbf{U} \in L^2(t_0 - R^2, t_0; \mathbf{W}^{1,2}_\sigma(B_R))$ *and* $q \in L^2(Q_R)$ *satisfy*

$$\begin{aligned}
&-\int_{Q_R} \mathbf{U}\cdot\boldsymbol{\psi}_t\,\mathrm{d}x\,\mathrm{d}t \,+\, \int_{Q_R} \nabla\mathbf{U} : \nabla\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{Q_R} q\,\mathrm{div}\,\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t \,+\, \int_{Q_R} \mathbf{f} : \nabla\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{14}$$

for all $\boldsymbol{\psi} \in C^\infty_c(\,]t_0 - R^2, t_0[; \mathbf{C}^\infty_c(B_R))$.
Then

$$\int_{Q_r} |\mathbf{U} - \mathbf{U}_{Q_r}|^2\,\mathrm{d}x\,\mathrm{d}t \,\le\, cr^{5+2\alpha} \quad \forall\, 0 < r \le R. \tag{15}$$

where $c = \mathrm{const}$ *does not depend on r.*

Proof We divide the proof into two steps.
1. The following Poincaré inequality holds:

$$\int_{Q_r} |\mathbf{U} - \mathbf{U}_{Q_r}|^2\,\mathrm{d}x\,\mathrm{d}t \,\le\, cr^2 \int_{Q_r} (|\nabla\mathbf{U}|^2 + (q - q_{B_r})^2 \,+\, |\mathbf{f} - \mathbf{f}_{Q_r}|^2)\,\mathrm{d}x\,\mathrm{d}t \tag{16}$$

for all $0 < r \le R$ ($c = \mathrm{const}$ independent of r). Indeed, by elementary arguments,

$$\int_{Q_r} |\mathbf{U} - \mathbf{U}_{Q_r}|^2\,\mathrm{d}x\,\mathrm{d}t \le cr^2 \int_{Q_r} |\nabla\mathbf{U}|^2\,\mathrm{d}x\,\mathrm{d}t \,+\, \int_{t_0-r^2}^{t_0}\int_{t_0-r^2}^{t_0} |\widetilde{\mathbf{U}}_{B_r}(t) - \widetilde{\mathbf{U}}_{B_r}(s)|^2\,\mathrm{d}s\,\mathrm{d}t.$$

Here

$$\tilde{\mathbf{U}}_{B_r}(t) := \left(\int_{B_r} \zeta^2\, \mathrm{d}x\right)^{-1} \int_{B_r} \mathbf{U}(y,t)\zeta^2(y)\, \mathrm{d}y \quad \text{for a.e.} \quad t \in]t_0 - r^2, t_0[,$$

where $\zeta \in C_c^\infty(B_r)$ is a cut-off function such that $\zeta = 1$ in $B_{r/2}$, and $0 \le \zeta \le 1$, $|\nabla\zeta| \le \dfrac{c_0}{r}$ in B_r ($c_0 = $ const). Now the difference $\widetilde{\mathbf{U}}_{B_r}(t) - \widetilde{\mathbf{U}}_{B_r}(s)$ can be estimated by using (14). This is readily done by following word by word the reasoning in [7].

2. In (14) we take $\boldsymbol{\varphi} \in C_c^\infty(\,]t_0 - R^2, t_0[\,; \mathbf{C}_{c,\sigma}^\infty(B_R))$. Then

$$-\int_{Q_R} \mathbf{U}\cdot\boldsymbol{\varphi}_t\, \mathrm{d}x\, \mathrm{d}t + \int_{Q_R} \nabla\mathbf{U} : \nabla\boldsymbol{\varphi}\, \mathrm{d}x\, \mathrm{d}t = \int_{Q_R} \mathbf{f} : \nabla\boldsymbol{\varphi}\, \mathrm{d}x\, \mathrm{d}t \tag{17}$$

(cf. (6)). Hence the assertion of Prop. 1 is true for $\mathbf{U}$. Observing (13) from (5) we obtain

$$\int_{Q_r} |\nabla\mathbf{U}|^2\, \mathrm{d}x\, \mathrm{d}t \le c r^{3+2\alpha} \quad \forall\, 0 < r \le R. \tag{18}$$

Next, the Corollary in Appendix A.2 gives

$$\int_{Q_r} (q - q_{B_r})^2\, \mathrm{d}x\, \mathrm{d}t \le c\left\{\left(\frac{r}{R}\right)^5 \int_{Q_R} (q - q_{B_R})^2\, \mathrm{d}x\, \mathrm{d}t + R^{3+2\alpha}\right\}$$

for all $0 < r \le R$. Thus,

$$\int_{Q_r} (q - q_{B_r})^2\, \mathrm{d}x\, \mathrm{d}t \le c r^{3+2\alpha} \quad \forall\, 0 < r \le R. \tag{19}$$

Inserting (18) and (19) into (16) and using once more (13) implies (15). □

4 Proof of the main theorem completed

As above, let $Q_{R_0} = B_{R_0}(x_0)\times\,]t_0 - R_0^2, t_0[$ satisfy $\overline{Q}_{R_0} \subset Q$. Define $\mathbf{F} := \nabla\mathbf{u} - \mathbf{f}$ a.e. in Q_{R_0}. Let $0 < R \le \dfrac{R_0}{2}$. Then

$$-\int_{Q_{2R}} \mathbf{u}\cdot\boldsymbol{\varphi}_t\, \mathrm{d}x\, \mathrm{d}t + \int_{Q_{2R}} \mathbf{F} : \nabla\boldsymbol{\varphi}\, \mathrm{d}x\, \mathrm{d}t = 0 \tag{20}$$

for all $\boldsymbol{\varphi} \in C_c^\infty(\,]t_0 - 4R^2, t_0[\,; \mathbf{C}_{c,\sigma}^\infty(B_{2R}))$ (cf. (6)). It follows that there exist functions

$$p_0 \in L^2(t_0 - 4R^2, t_0; A^2(B_{2R})), \quad p_h \in C_w([t_0 - 4R^2, t_0]; B^2(B_{2R}))$$

such that

$$\begin{cases} p_h \in C([t_0 - 4R^2, t_0]; W^{m,2}(B_{r/2^{m-1}})) \\ \textit{for every} \quad 0 < r < 2R \quad \textit{and} \quad m = 1, 2, \ldots, \end{cases} \tag{21}$$

$$\begin{cases} -\displaystyle\int_{Q_{2R}} (\mathbf{u} + \nabla p_h)\cdot\boldsymbol{\psi}_t\, \mathrm{d}x\, \mathrm{d}t + \int_{Q_{2R}} \mathbf{F} : \nabla\boldsymbol{\psi}\, \mathrm{d}x\, \mathrm{d}t = \int_{Q_{2R}} p_0 \mathrm{div}\,\boldsymbol{\psi}\, \mathrm{d}x\, \mathrm{d}t \\ \textit{for all} \quad \boldsymbol{\psi} \in C_c^\infty(\,]t_0 - 4R^2, t_0[; \mathbf{C}_c^\infty(B_{2R})) \end{cases} \tag{22}$$

(see Appendix A.1 and A.3).

Define $\mathbf{U} := \mathbf{u} + \nabla p_h$ a.e. in Q_R. Then $\mathbf{U} \in L^2(t_0 - R^2, t_0; \mathbf{W}^{1,2}_\sigma(B_R))$ and

$$-\int_{Q_R} \mathbf{U} \cdot \boldsymbol{\psi}_t \, \mathrm{d}x \, \mathrm{d}t + \int_{Q_R} \nabla \mathbf{U} : \nabla \boldsymbol{\psi} \, \mathrm{d}x \, \mathrm{d}t = \int_{Q_R} p_0 \mathrm{div} \, \boldsymbol{\psi} \, \mathrm{d}x \, \mathrm{d}t + \int_{Q_R} \mathbf{f} : \nabla \boldsymbol{\psi} \, \mathrm{d}x \, \mathrm{d}t$$

for all $\boldsymbol{\psi} \in C^\infty_c(\,]t_0 - R^2, t_0[; \mathbf{C}^\infty_c(B_R))$. By Prop. 2 (Sect. 3),

$$\int_{Q_r} |\mathbf{U} - \mathbf{U}_{Q_r}|^2 \, \mathrm{d}x \, \mathrm{d}t \le r^{5+2\alpha} \quad \forall \, 0 < r \le R. \tag{23}$$

Now, fix any domain $\Omega' \subset \Omega$ such that $\overline{\Omega}' \subset \Omega$, and fix $0 < t' < T$. Then (23) holds for all $R < \frac{R_0}{2}$ where $\overline{Q}_{R_0} \subset \overline{\Omega'} \times [t', T]$ (with $c = $ const independent of R). By a theorem of Da Prato [3] (see also [1]), (23) implies: for every compact set $\mathcal{K} \subset Q' = \Omega' \times]t', T[$ there exists a constant $c = c_{\mathcal{K}}$ such that

$$|\mathbf{U}(x, s) - \mathbf{U}(y, t)| \le c(|x - y|^\alpha + |s - t|^{\alpha/2}) \quad \forall \, (x, s), (y, t) \in \mathcal{K}.$$

Finally, take $m = 4$ in (21). Then $\nabla p_h(\cdot, t) \in \mathbf{C}^1(\overline{B}_{r/2^3})$ for all $t \in [t_0 - 4R^2, t_0]$. It follows

$$\mathbf{u} \in C(\,]t', T[; \mathbf{C}^\alpha(\overline{\Omega}')).$$

Appendix

In what follows, let G be a bounded domain in $\mathbb{R}^n$ ($n \ge 2$).

A.1 *An orthogonal decomposition of $L^2(G)$*

Define

$$W^{2,2}_0(G) := \textit{closure of} \quad C^\infty_c(G) \quad \textit{in } W^{2,2}(G)$$

By integration by parts,

$$\sum_{i,j=1}^n \int_G \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \, \mathrm{d}x = \int_G (\Delta u) \Delta v \, \mathrm{d}x \quad \forall \, u, v \in W^{2,2}_0(G).$$

It follows that $W^{2,2}_0(G)$ is a Hilbert space with respect to the scalar product

$$(u, v)_\Delta := \int_G (\Delta u) \Delta v \, \mathrm{d}x.$$

The following result is well-known.

Theorem *There holds*

$$L^2(G) = A^2(G) \oplus B^2(G) \quad \textit{(orthogonal decomposition)},$$

where

$$A^2(G) := \Big\{ q_0 \in L^2(G); \quad q_0 = \Delta z, \ z \in W^{2,2}_0(G) \Big\},$$
$$B^2(G) := \Big\{ q_h \in L^2(G); \quad \int_G q_h \Delta \varphi \, \mathrm{d}x = 0 \ \ \forall \varphi \in C^\infty_c(G) \Big\}.$$

Thus, for every $q \in L^2(G)$, we have $q = q_0 + q_h$ where $q_0 \in A^2(G)$, $q_h \in B^2(G)$ and $(q_0, q_h)_{L^2} = 0$. By a well-known lemma of Weyl, the function q_h is harmonic in G.

A.2 *A Campanato-type inequality*

Proposition *Let $B_R = B_R(x_0)$ be a ball in $\mathbb{R}^n$. Let $\mathbf{F} \in [L^2(B_R)]^{n^2}$ and $q \in L^2(B_R)$ satisfy*

$$\int_{B_R} \mathbf{F} : \nabla^2 \varphi \, \mathrm{d}x \;=\; \int_{B_R} q \Delta\varphi \, \mathrm{d}x \quad \forall \varphi \in C_c^\infty(B_R).$$

Then, for all $0 < r \le R$ and all $\mathbf{\Lambda} \in \mathbb{R}^{n^2}$,

$$\int_{B_r} (q - q_{B_r})^2 \, \mathrm{d}x \;\le\; c\left\{ \left(\frac{r}{R}\right)^{n+2} \int_{B_R} (q - q_{B_R})^2 \, \mathrm{d}x \;+\; \|\mathbf{F} - \mathbf{\Lambda}\|^2_{[L^2(B_R)]^{n^2}} \right\},$$

where the constant c does not depend on $r (B_r = B_r(x_0))$.

Proof Let $q = q_0 + q_h$ denote the decomposition according to A.1. We have

$$\|q_0\|_{L^2(B_R)} \;=\; \sup_{\varphi \in C_c^\infty(B_R), \|\Delta\varphi\|_{L^2} \le 1} \left| \int_{B_R} q_0 \Delta\varphi \, \mathrm{d}x \right| \;\le\; c\|\mathbf{F} - \mathbf{\Lambda}\|_{[L^2(B_R)]^{n^2}}.$$

On the other hand, q_h being harmonic in B_R there holds

$$\int_{B_r} (q_h - q_{h,B_r})^2 \, \mathrm{d}x \;\le\; c\left(\frac{r}{R}\right)^{n+2} \int_{B_R} (q_h - q_{h,B_R})^2 \, \mathrm{d}x \quad \forall\, 0 < r \le R.$$

Thus,

$$\begin{aligned} \int_{B_r} (q - q_{B_r})^2 \, \mathrm{d}x &\le 2\int_{B_r} q_0^2 \, \mathrm{d}x \;+\; 2\int_{B_r} (q_h - q_{h,B_r})^2 \, \mathrm{d}x \\ &\le c\left(\|\mathbf{F} - \mathbf{\Lambda}\|^2_{[L^2(B_R)]^{n^2}} \;+\; \left(\frac{r}{R}\right)^{n+2} \int_{B_R} (q_h - q_{h,B_R})^2 \, \mathrm{d}x \right) \end{aligned}$$

for all $0 < r \le R$.

Whence the claim.

Corollary *Let $\mathbf{f} \in [L^2(Q_R)]^{n^2}$. Assume $\mathbf{U} \in L^2(t_0 - R^2, t_0; [W^{1,2}_\sigma(B_R)]^n)$ and $q \in L^2(Q_R)$ satisfy*

$$\begin{aligned} -\int_{Q_R} \mathbf{U} \cdot \boldsymbol{\psi}_t \, \mathrm{d}x \, \mathrm{d}t \;&+\; \int_{Q_R} \nabla \mathbf{U} : \nabla \boldsymbol{\psi} \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_{Q_R} q \operatorname{div} \boldsymbol{\psi} \, \mathrm{d}x \, \mathrm{d}t \;+\; \int_{Q_R} \mathbf{f} : \nabla \boldsymbol{\psi} \, \mathrm{d}x \, \mathrm{d}t \end{aligned}$$

for all $\boldsymbol{\psi} \in C_c^\infty(\,]t_0 - R^2, t_0[\,; [C_c^\infty(B_R)]^n)$.

Then, for all $0 < r \le R$ and all $\mathbf{\Lambda} \in \mathbb{R}^{n^2}$,

$$\int_{Q_r} (q - q_{B_r}(t))^2 \, \mathrm{d}x \, \mathrm{d}t \;\le\; c\left\{ \left(\frac{r}{R}\right)^{n+2} \int_{Q_R} (q - q_{B_R}(t))^2 \, \mathrm{d}x \, \mathrm{d}t \;+\; \|\mathbf{f} - \mathbf{\Lambda}\|^2_{L^2(Q_R; \mathbb{R}^{n^2})} \right\}$$

where the constant c does not depend on $r (Q_r = B_r \times \,]t_0 - r^2, t_0[\,)$.

Proof Let $\phi \in C_c^\infty(B_R)$ and $\eta \in C_c^\infty(\,]t_0 - R^2, t_0[\,)$. With test function $\boldsymbol{\psi}(x,t) := (\nabla\phi(x))\eta(t)$, $(x,t) \in Q_R$, we obtain

$$\begin{aligned} -\int_{Q_R} \mathbf{U} \cdot \nabla\phi\eta' \, \mathrm{d}x \, \mathrm{d}t \;&+\; \int_{Q_R} \nabla \mathbf{U} : \nabla^2\phi\eta \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_{Q_R} q \Delta\phi\eta \, \mathrm{d}x \, \mathrm{d}t \;+\; \int_{Q_R} \mathbf{f} : \nabla^2\phi\eta \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

Observing that

$$\int_{B_R} \mathbf{U}(x,t)\cdot\nabla\phi(x)\,\mathrm{d}x = 0, \quad \int_{B_R} \nabla\mathbf{U}(x,t) : \nabla^2\phi(x)\,\mathrm{d}x = 0 \quad (i = 1,\ldots,n)$$

for a.e. $t \in\,]t_0 - R^2, t_0[$, it follows

$$\int_{Q_R} q\Delta\phi\eta\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_R} \mathbf{f} : \nabla^2\phi\eta\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Hence

$$\int_{B_R} q(x,t)\Delta\phi(x)\,\mathrm{d}x + \int_{B_R} \mathbf{f}(x,t) : \nabla^2\phi(x)\,\mathrm{d}x = 0 \quad \text{for a.e.} \quad t \in\,]t_0 - R^2, t_0[$$

where the null set for which this equation fails, does not depend on ϕ. Applying the Proposition gives

$$\begin{aligned}&\int_{B_r} (q(x,t) - q_{B_r}(t))^2\,\mathrm{d}x \\ &\quad \le c\left\{\left(\frac{r}{R}\right)^{n+2} \int_{B_R} (q(x,t) - q_{B_R}(t))^2\,\mathrm{d}x + \|\mathbf{f}(t) - \mathbf{\Lambda}\|^2_{[L^2(B_R)]^{n^2}}\right\}\end{aligned}$$

for all $0 < r \le R$. Integration over $]t_0 - R^2, t_0[$ completes the proof.

A.3 *Existence and decomposition of a pressure*

The following result is the L^2-variant of a more general theorem proved in [13].

Theorem *Let* $\mathbf{F} \in L^2(a,b;[L^2(G)]^{n^2})(-\infty<a<b<+\infty)$. *Let* $\mathbf{u} \in C_w([a,b];[L^2(G)]^n)$ *satisfy*

$$\begin{aligned}&\int_G \mathbf{u}(x,t)\cdot\nabla\psi(x)\,\mathrm{d}x = 0 \quad \forall\, t \in [a,b], \quad \forall\,\psi \in C_c^\infty(G), \\ &-\int_a^b \int_G \mathbf{u}\cdot\boldsymbol{\varphi}_t\,\mathrm{d}x\,\mathrm{d}t + \int_a^b\int_G \mathbf{F} : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t = 0 \;\; \forall\,\boldsymbol{\varphi} \in C_c^\infty(\,]a,b[\,;\mathbf{C}^\infty_{c,\sigma}(G)).\end{aligned}$$

Then, there exist

$$p_0 \in L^2(a,b;A^2(G)), \quad p_h \in C_w([a,b];B^2(G))$$

such that: for any given (fixed) domain $G' \subset \overline{G'} \subset G$ *and any given (fixed)* $m \in \mathbb{N}$ *there holds*

$$\begin{aligned}&p_h \in C([a,b];W^{m,2}(G')),\; \|p_h\|_{C([a,b];W^{m,2}(G'))} \le c\|p_h\|_{L^\infty(a,b;L^2(G))}, \\ &\quad -\int_a^b\int_G (\mathbf{u} + \nabla p_h)\cdot\boldsymbol{\psi}_t\,\mathrm{d}x\,\mathrm{d}t + \int_a^b\int_G \mathbf{F} : \nabla\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_G \mathbf{u}(x,a)\cdot\boldsymbol{\psi}(x,a)\,\mathrm{d}x + \int_a^b\int_G p_0 \mathrm{div}\,\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for all $\boldsymbol{\psi} \in C_c^\infty([a,b[;[C_c^\infty(G)]^n)$.

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